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MICROSTABILITY THEORY FOR DISTRIBUTIONS SEPARABLE IN ENERGY AND PITCH ANGLE*

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Abstract: Distribution functions that are separable in energy and pitch angle allow analytical calculation of one or two velocity-space integrals that appear in the linear theory of certain microinstabilities.

1. Introduction

Linear stability theories of high-frequency waves in magnetized plasmas have usually relied on model particle distribution functions f. An oft-used model is

$$f = v_\perp^{2\ell} \exp(-\alpha_\perp v_\perp^2 - \alpha_\parallel v_\parallel^2) \quad , \quad \ell = 0,1,2, \cdots \tag{1}$$

which is separable in the perpendicular and parallel velocity components $v_\perp$ and $v_\parallel$: $f = f_\perp(v_\perp)\ f_\parallel(v_\parallel)$. (Normalization constants in f are omitted in this paper.) This paper presents an additional class of model distributions that provide more realistic descriptions of some plasmas than does Eq. (1) or, more generally, $f = f_\perp f_\parallel$.

Model distributions allow one or two velocity-space integrations to be performed analytically, which greatly reduces the numerical work required to solve a dispersion relation. For example, for circularly polarized waves propagating along the ambient magnetic field ($k_\perp = 0$) in a nonrelativistic plasma, one must calculate double velocity-space integrals like those in the susceptibility

$$\chi = \frac{\omega}{\omega - \Omega} + 2\pi \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty dv_\perp v_\perp f \left[\frac{k_\parallel^2\ v_\perp^2}{2(\omega - \Omega - k_\parallel v_\parallel)^2} + \left(\frac{\Omega}{\omega - \Omega}\right) \frac{k_\parallel v_\parallel}{\omega - \Omega - k_\parallel v_\parallel} \right] \quad . \tag{2}$$

For the distribution (1),

$$\chi = \frac{\omega}{\omega - \Omega} - \left[\frac{(\ell + 1)\alpha_{\parallel}}{\alpha_{\perp}} + \frac{\Omega}{\omega - \Omega}\right] Z_1(\zeta) \quad , \tag{3}$$

$\zeta \equiv (\omega - \Omega)\alpha_{\parallel}^{1/2}/k_{\parallel}$, and Z_1 is the $m = 1$ member of a family of functions ($m = 0,1,2, \cdots$)

$$Z_m(\zeta) \equiv \pi^{-1/2} \int_{-\infty}^{\infty} du \; (u - \zeta)^{-1} \, u^m \exp(-u^2) \quad , \tag{4}$$

which are related to the standard plasma dispersion function Z_0. Numerical calculation of the right side of Eq. (3) involves evaluation of the analytic function Z_1, which is orders of magnitude quicker than the evaluation of the double integral in Eq. (2).

2. Nonrelativistic Plasmas with Symmetric Pitch-Angle Distributions

Consider now the class of distributions that are separable in speed v and pitch angle $\phi = \cos^{-1}(v_{\parallel}/v)$: $f = F(v)G(\phi)$. For nonrelativistic plasma and $k_{\perp} = 0$, the class of speed distributions

$$F(v) = v^{2\ell} \exp(-\alpha v^2) \quad , \quad \ell = 0,1,2, \cdots \tag{5}$$

allows one of the two velocity-space integrals ($\int_0^{\infty} dv \; v^2 \cdots$) in χ to be performed analytically. The pitch-angle integral can also be performed analytically if we choose

$$G = [H(\phi-\phi_+) - H(\phi-\phi_-) + H(\phi-\pi+\phi_-) - H(\phi-\pi+\phi_+)] \quad , \tag{6}$$

where H denotes the Heaviside step function. This $G(\phi)$, which obeys the symmetry $G(\phi) = G(\pi - \phi)$, is illustrated in Fig. 1. For $\phi_- = \pi/2$, $G(\phi)$ represents a distribution of

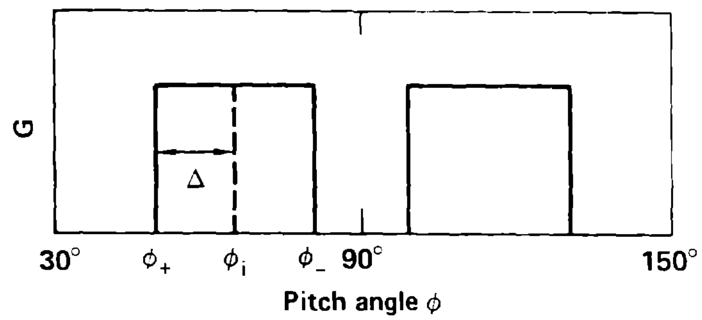

Fig. 1. The symmetric pitch-angle distribution given by Eq. (6).

particles with pitch angles in the range $\phi_+ < \phi < \pi - \phi_+$ and models confinement in a magnetic mirror of ratio $(\sin \phi_+)^{-2}$. For $\phi_- < \pi/2$, $G(\phi)$ represents particles with pitch angles near $\phi = \phi_i = (\phi_+ + \phi_-)/2$ and $\phi = \pi - \phi_i$, i.e., near a conical surface in velocity space. For the distribution given by Eqs. (5)-(6), the susceptibility (2) is

$$\chi = \frac{\omega}{\omega - \Omega} + \frac{\chi_+ \cos \phi_+ - \chi_- \cos \phi_-}{\cos \phi_+ - \cos \phi_-}$$

$$\chi_\pm = - \frac{1}{2\left(\ell+\frac{1}{2}\right)!} \left[\tan^2\phi_\pm \, Z_{2\ell+3}(\zeta_\pm) + \frac{\omega}{\omega-\Omega} \, \ell! \sum_{n=0}^{\ell} \frac{1}{n!} Z_{2n+1}(\zeta_\pm)\right]$$

$$\zeta_\pm \equiv (\omega - \Omega) \, \alpha^{1/2}/k_{\parallel} \cos \phi_\pm \quad .$$

3. Nonrelativistic Plasmas with Unsymmetric Pitch-Angle Distributions

Pitch-angle distributions $G(\phi)$ that are not symmetric about $\phi = \pi/2$ but are piecewise constant in ϕ also allow the pitch-angle integral to be performed analytically. An example is $G(\phi) = H(\phi - \theta)$, $0 \le \phi \le \pi$, which represents a particle distribution that is isotropic except for the absence of all particles in an unsymmetric loss cone $0 \le \phi \le \theta$. For nonrelativistic plasma, for $k_\perp = 0$, and for the Maxwellian speed distribution (5) with $\ell = 0$, the susceptibility (2) is

$$\chi = \frac{\omega}{\omega - \Omega} \left(1 - \frac{1}{(1 + \cos \theta)} \left\{ Z_1(\zeta_0) - \bar{Z}_1(\zeta_0) + [Z_1(\zeta) + \bar{Z}_1(\zeta)] \cos \theta \right\}\right) - \frac{\sin \theta \tan \theta}{(1 + \cos \theta)} [Z_3(\zeta) + \bar{Z}_3(\zeta)]$$

$$\zeta_0 \equiv (\omega - \Omega) \, \alpha^{1/2}/k_{\parallel} \quad , \quad \zeta \equiv \zeta_0 \sec \theta \quad .$$

The functions $\bar{Z}_1$ and $\bar{Z}_3$ are defined by $(m = 0,1,2, \cdots)$

$$\bar{Z}_m(\zeta) \equiv \pi^{-1/2} \int_{-\infty}^{\infty} du \, \mathrm{sgn}(u) \, (u - \zeta)^{-1} \, u^m \exp(-u^2) \quad ,$$

which differs from Eq. (4) only through the factor of sgn(u). Recursion relations exist among the $\bar{Z}_m$, and

$$\bar{Z}_0(\zeta) = \pi^{-1/2} \exp(-\zeta^2) E_1(-\zeta^2) \quad ,$$

where E_1 is the exponential-integral function.

4. Relativistic Plasmas

Plasmas with substantial fractions of relativistic electrons require the numerical calculation of one or two velocity-space integrals that appear in dispersion relations. This difficulty arises because of the complicated dependence on velocity-space variables of the wave-particle resonance conditions

$$\omega\gamma - n\Omega - k_{\parallel}p_{\parallel}/M = 0 \quad , \tag{7}$$

where n is an integer, $\gamma = [1 + (p_{\perp}^2 + p_{\parallel}^2)/M^2c^2]^{1/2}$, $p_{\perp}$ and $p_{\parallel}$ are momentum components, and M is the rest mass. One velocity-space integral (the pitch-angle integral) can be performed analytically if either $k_{\perp} = 0$ or $k_{\parallel} = 0$, provided the distribution f has the form $F(v)G(\phi)$ and $G(\phi)$ is piecewise constant [e.g., Eq. (6)]. Since the speed integral is evaluated numerically, any $F(v)$ can be used. Convenient velocity-space variables are $x = \cos \phi$ and $y = \gamma^{-1}$, in terms of which Eq. (7) becomes

$$\omega - n\Omega y - k_{\parallel}cx\ (1 - y^2)^{1/2} = 0 \quad . \tag{8}$$

The x dependence in Eq. (8) is simple enough to allow analytical calculation of the x (pitch-angle) integral if $k_{\perp} = 0$. For the other special wave-propagation direction ($k_{\parallel} = 0$), Eq. (8) has no x dependence but the Bessel functions that appear for $k_{\perp} \neq 0$ cause x dependence. Still, the x integral can be replaced by a rapidly convergent infinite sum and, in this sense, can be performed analytically.

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